KONKRETNA MATEMATIKA 2
Zadaća 1

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Zadatak 1. Izračunajte $\left\{ {2007 \atop 4} \right\}$ (Stirlingov broj druge vrste).

Rješenje. Koristeći rekurziju $\left\{ {n \atop k} \right\} = k\left\{ {n-1 \atop k} \right\} + \left\{ {n-1 \atop k-1} \right\}$, izračunajmo $\left\{ {n \atop 4} \right\}$ za $n \geqslant 4$:

$$\begin{aligned}
\left\{ {n \atop 4} \right\} &= 4\left\{ {n-1 \atop 4} \right\} + \left\{ {n-1 \atop 3} \right\} \\
&= 4^2\left\{ {n-2 \atop 4} \right\} + 4\left\{ {n-2 \atop 3} \right\} + \left\{ {n-1 \atop 3} \right\} \\
&= 4^3\left\{ {n-3 \atop 4} \right\} + 4^2\left\{ {n-3 \atop 3} \right\} + 4\left\{ {n-2 \atop 3} \right\} + \left\{ {n-1 \atop 3} \right\} \\
&\vdots \\
&= 4^{n-4}\left\{ {4 \atop 4} \right\} + \sum_{i=4}^{n-1} 4^{n-1-i}\left\{ {i \atop 3} \right\} \\
&= \sum_{i=3}^{n-1} 4^{n-1-i}\left\{ {i \atop 3} \right\} \\
&= 4^{n-4}\sum_{i=0}^{n-4} 4^{-i}\left\{ {i+3 \atop 3} \right\}
\end{aligned}$$

Na sasvim analogan način izračunamo $\left\{ {n \atop 3} \right\}$ za $n \geqslant 3$. Iskoristimo i $\left\{ {n \atop 2} \right\} = 2^{n-1}-1,\ n \geqslant 2$ da dobijemo:

$$\begin{aligned}
\left\{ {n \atop 3} \right\} &= 3^{n-3}\sum_{j=0}^{n-3} 3^{-j}\left\{ {j+2 \atop 2} \right\} \\
&= 3^{n-3}\sum_{j=0}^{n-3} 3^{-j}\left(2^{j+1}-1\right) \\
&= 3^{n-3}\left(2\sum_{j=0}^{n-3}\left(\frac{2}{3}\right)^j - \sum_{j=0}^{n-3}\left(\frac{1}{3}\right)^j\right) \\
&= 3^{n-3}\left(2\cdot\frac{1-\left(\frac{2}{3}\right)^{n-2}}{\frac{1}{3}} - \frac{1-\left(\frac{1}{3}\right)^{n-2}}{\frac{2}{3}}\right) \\
&= \frac{1}{2}\left(3^{n-1}+1\right) - 2^{n-1}
\end{aligned}$$

Sad se možemo vratiti na $\left\{ {n \atop 4} \right\}$:

$$\begin{aligned}
\left\{ {n \atop 4} \right\} &= 4^{n-4} \sum_{i=0}^{n-4} 4^{-i} \left(\frac{1}{2} \left(3^{i+2} + 2 \right) - 2^{i+2} \right) \\
&= 4^{n-4} \left(\frac{9}{2} \sum_{i=0}^{n-4} \left(\frac{3}{4} \right)^i + \frac{1}{2} \sum_{i=0}^{n-4} \left(\frac{1}{4} \right)^i - 4 \sum_{i=0}^{n-4} \left(\frac{1}{2} \right)^i \right) \\
&= 4^{n-4} \left(\frac{9}{2} \cdot \frac{1 - \left(\frac{3}{4} \right)^{n-3}}{\frac{1}{4}} + \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{4} \right)^{n-3}}{\frac{3}{4}} - 4 \cdot \frac{1 - \left(\frac{1}{2} \right)^{n-3}}{\frac{1}{2}} \right) \\
&= \frac{1}{24} \left(4^n - 4 \cdot 3^n + 6 \cdot 2^n - 4 \right)
\end{aligned}$$

Preostaje uvrstiti $n = 2007$:

$$\left\{ {2007 \atop 4} \right\} = \frac{1}{24} \left(4^{2007} - 4 \cdot 3^{2007} + 6 \cdot 2^{2007} - 4 \right) .$$

□

Zadatak 2. Odredite hipergeometrijsku funkciju $F(z)$ koja zadovoljava diferencijalnu jednadžbu

$$zF'(z) + F(z) = \frac{1}{1-z} .$$

Rješenje. Pretpostavimo da formalan red $F(z) = \sum_{k \geqslant 0} t_k z^k$ zadovoljava zadanu diferencijalnu jednadžbu. Tada imamo:

$$z \sum_{k \geqslant 0} k t_k z^{k-1} + \sum_{k \geqslant 0} t_k z^k = \sum_{k \geqslant 0} z^k ,$$

tj.

$$\sum_{k \geqslant 0} (k+1) t_k z^k = \sum_{k \geqslant 0} z^k .$$

Prema tome, slijedi da mora biti $(k+1)t_k = 1$, tj. $t_k = \frac{1}{k+1}$. Računamo:

$$\begin{aligned}
t_0 &= 1 \\
\frac{t_{k+1}}{t_k} &= \frac{k+1}{k+2} = \frac{(k+1)(k+1)}{k+2} \cdot \frac{1}{k+1}
\end{aligned}$$

Zaključujemo:

$$F(z) = F\left({1,1 \atop 2} \middle| \, z \right) .$$

□

Zadatak 3. Koristeći Gosperov algoritam izračunajte sumu

$$\sum_{k=1}^{n} \frac{2^{2k-1}}{k} \binom{2k}{k}^{-1} .$$

Rješenje. Označimo $t(k) := \frac{2^{2k-1}}{k}\binom{2k}{k}^{-1}$ i pronađimo $T(k)$ tako da bude $t(k) = T(k+1) - T(k)$:

$$\begin{aligned}\frac{t(k+1)}{t(k)} &= \frac{2^{2k+1}}{k+1} \cdot \frac{(k+1)!(k+1)!}{(2k+2)!} \cdot \frac{k}{2^{2k-1}} \cdot \frac{(2k)!}{k!k!} \\ &= \frac{4k(k+1)}{(2k+1)(2k+2)} = \frac{k}{k+\frac{1}{2}}\end{aligned}$$

Definiramo $p(k) := 1$, $q(k) := k$ i $r(k) := k - \frac{1}{2}$. Zadovoljeno je:

- $\dfrac{t(k+1)}{t(k)} = \dfrac{p(k+1)q(k)}{p(k)r(k+1)}$
- $(k+\alpha)\backslash q(k) \ \wedge \ (k+\beta)\backslash r(k) \implies \alpha - \beta \notin \mathbb{N}$

Tražimo $T(k)$ u obliku $T(k) = \frac{r(k)s(k)t(k)}{p(k)}$. Poznato nam je da je $s(k)$ polinom. Označimo $d := \deg(s)$ i definiramo $Q(k) := q(k) - r(k) = \frac{1}{2}$ i $R(k) := q(k) + r(k) = 2k - \frac{1}{2}$. Budući da je $\deg(Q) < \deg(R)$, imamo dva moguća slučaja od kojih u prvom vrijedi $d = \deg(p) - \deg(R) + 1 = 0 - 1 + 1 = 0$. U skladu s tim, stavimo da je $s(k) = c$ i uvrstimo to u rekurziju:

$$\begin{aligned}2p(k) &= Q(k)(s(k+1) + s(k)) + R(k)(s(k+1) - s(k)), \text{ tj.} \\ 2 &= \frac{1}{2} \cdot 2c\end{aligned}$$

Dakle, $s(k) = 2$. Slijedi da je

$$\begin{aligned}T(k) &= \frac{r(k)s(k)t(k)}{p(k)} \\ &= 2^{2k}\frac{k-\frac{1}{2}}{k}\binom{2k}{k}^{-1} \\ &= 4^{k-1}\frac{4(k-\frac{1}{2})}{k} \cdot \frac{k!k!}{(2k)(2k-1)(2k-2)!} \\ &= 4^{k-1}\frac{(k-1)!(k-1)!}{(2k-2)!} \\ &= 4^{k-1}\binom{2(k-1)}{k-1}^{-1}\end{aligned}$$

Sad lako izračunamo:

$$\begin{aligned}\sum_{k=1}^{n} \frac{2^{2k-1}}{k}\binom{2k}{k}^{-1} &= T(k)\Big|_{1}^{n+1} \\ &= 4^n\binom{2n}{n}^{-1} - 1\end{aligned}$$

□