KONKRETNA MATEMATIKA 2
Zadaća 2

Filip Nikšić
fniksic@gmail.com

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1. Dokažite jednakost hipergeometrijskih funkcija

$$F\left(\begin{matrix}a, b\\ c\end{matrix}\,\middle|\, z\right) = (1-z)^{c-a-b} F\left(\begin{matrix}c-a, c-b\\ c\end{matrix}\,\middle|\, z\right).$$

Rješenje. Dokažimo prvo Pfaffov identitet:

$$(1-z)^{-a} F\left(\begin{matrix}a, b\\ c\end{matrix}\,\middle|\, \frac{-z}{1-z}\right) = F\left(\begin{matrix}a, c-b\\ c\end{matrix}\,\middle|\, z\right).$$

Naime, lijevu stranu možemo raspisati na sljedeći način:

$$\begin{aligned}
(1-z)^{-a} F\left(\begin{matrix}a, b\\ c\end{matrix}\,\middle|\, \frac{-z}{1-z}\right) &= \sum_{k\geqslant 0} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} (-z)^k (1-z)^{-a-k} \\
&= \sum_{k\geqslant 0} \frac{a^{\overline{k}} b^{\overline{k}}}{c^{\overline{k}} k!} (-z)^k \sum_{j\geqslant 0} \binom{j+a+k-1}{j} z^j \\
&= \sum_{n\geqslant 0} \Bigg(\sum_{k\geqslant 0} \underbrace{\frac{a^{\overline{k}} b^{\overline{k}} (-1)^k}{c^{\overline{k}} k!} \binom{n+a-1}{n-k}}_{t_k} \Bigg) z^n
\end{aligned}$$

Za koeficijent uz z^n vrijedi:

$$\begin{aligned}
t_0 &= \binom{n+a-1}{n} \\
\frac{t_{k+1}}{t_k} &= \frac{a^{\overline{k+1}} b^{\overline{k+1}} (-1)^{k+1} (n+a-1)^{\underline{n-k-1}}}{c^{\overline{k+1}} (k+1)! (n-k-1)!} \cdot \frac{c^{\overline{k}} k! (n-k)!}{a^{\overline{k}} b^{\overline{k}} (-1)^k (n+a-1)^{\underline{n-k}}} \\
&= \frac{(k+a)(k+b)(k-n)}{(k+c)(k+1)(k+a)}
\end{aligned}$$

Zaključujemo da je koeficijent uz z^n jednak (iskoristimo hipergeometrijski oblik Vandermondeove konvolucije)

$$\begin{aligned}
\binom{n+a-1}{n} F\left(\begin{matrix}b, -n\\ c\end{matrix}\,\middle|\, 1\right) &= \frac{(n+a-1)^{\underline{n}} (c-b)^{\overline{n}}}{n! c^{\overline{n}}} \\
&= \frac{a^{\overline{n}} (c-b)^{\overline{n}}}{c^{\overline{n}} n!}
\end{aligned}$$

Prema tome,

$$(1-z)^{-a} F\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, \frac{-z}{1-z}\right) = F\left(\begin{matrix} a, c-b \\ c \end{matrix} \,\middle|\, z\right).$$

Sad dvaput iskoristimo Pfaffov identitet da dokažemo traženu jednakost:

$$\begin{aligned}(1-z)^{c-a-b} F\left(\begin{matrix} c-a, c-b \\ c \end{matrix} \,\middle|\, z\right) &= (1-z)^{-b} F\left(\begin{matrix} c-a, b \\ c \end{matrix} \,\middle|\, \frac{-z}{1-z}\right) \\ &= F\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, z\right)\end{aligned}$$

□

2. Odredite funkciju izvodnicu za niz $(a_n)_{n\in\mathbb{Z}_+}$ zadan rekurzivno:

$$a_0 := 1, \quad a_1 := 1, \quad a_n := 1 + a_{n-1} + 2a_{n-2} + 2a_{n-3} + \cdots + 2a_0, \quad n \geqslant 2.$$

Rješenje. Primijetimo prvo da za $n \geqslant 2$ vrijedi:

$$\begin{aligned}a_{n+1} - a_n &= 1 + a_n + 2a_{n-1} + 2a_{n-2} + \cdots + 2a_0 - (1 + a_{n-1} + 2a_{n-2} + \cdots + 2a_0) \\ &= a_n + a_{n-1}\end{aligned}$$

Prema tome, rekurziju možemo zapisati na ekvivalentan način:

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 4, \quad a_n = 2a_{n-1} + a_{n-2}, \quad n \geqslant 3.$$

Ako pretpostavimo da je $a_n = 0$ za $n < 0$, možemo pisati:

$$a_n = 2a_{n-1} + a_{n-2} + [n=0] - [n=1] + [n=2], \quad \text{cijeli } n.$$

Pomnožimo obje strane sa z^n i sumirajmo po n:

$$\begin{aligned}\sum_n a_n z^n &= 2z \sum_n a_n z^n + z^2 \sum_n a_n z^n + \sum_{n=0} z^n - \sum_{n=1} z^n + \sum_{n=2} z^n \\ G(z) &= 2zG(z) + z^2 G(z) + 1 - z + z^2 \\ G(z) &= \frac{1 - z + z^2}{1 - 2z - z^2} \\ &= -1 + \frac{A}{1 - \xi_1 z} + \frac{B}{1 - \xi_2 z}\end{aligned}$$

Tu smo rastavom na parcijalne razlomke dobili brojeve:

$$\xi_1 = 1 + \sqrt{2} \qquad \xi_2 = 1 - \sqrt{2}$$

$$A = \frac{2\xi_1 - 3}{\xi_1 - \xi_2} \qquad B = \frac{2\xi_2 - 3}{\xi_2 - \xi_1}$$

Razvojem $G(z)$ u red potencija dobivamo:

$$\begin{aligned}G(z) &= -1 + A \sum_{n\geqslant 0} \xi_1^n z^n + B \sum_{n\geqslant 0} \xi_2^n z^n \\ &= \sum_{n\geqslant 0} \big(A\xi_1^n + B\xi_2^n - [n=0]\big) z^n\end{aligned}$$

Sad imamo jednostavan izraz za članove niza:

$$a_n = A\xi_1^n + B\xi_2^n - [n=0], \quad \text{cijeli } n \geqslant 0.$$

□